Journal of Engineering Mathematics, Vol. 3, No. 1, January 1969
Wolters-Noordhoff Publishing – Groningen
Printed in the Netherlands

Non-Linear Dispersion of Stokes Waves

H. W. HOOGSTRATEN

Department of Mathematics, Technological University, Delft, the Netherlands

(Received August 15, 1968)

SUMMARY
Asymptotic solutions representing slowly varying wavetrains are obtained for two-dimensional irrotational surface waves of finite amplitude on water of finite depth (Stokes waves) by means of a formal perturbation procedure. The resulting partial differential equations for wavenumber, frequency, amplitude, mean waveheight and mean velocity are identical with those found by Whitham [1], [2] using the averaged Lagrangian density method.

1. Introduction

Asymptotic solutions in the form of slowly varying wavetrains, i.e. waves for which the wavenumber, frequency, amplitude, etc. change only by a very small fraction of themselves within one period or one wavelength, have been obtained recently for Stokes waves by Whitham [1], [2] who used his averaged Lagrangian density method. In this paper we will derive the same results (a set of pertial differential equations for wavenumber, frequency, amplitude, mean waveheight and mean velocity) by means of an asymptotic representation of slowly varying wavetrains, which has proved to be also successful in the case of non-linear shallow water waves (Hoogstraten [3]).

The general formulation of the boundary value problem for the velocity potential of two-dimensional irrotational surface waves of finite amplitude on water of finite depth (Stokes waves) is as follows: within the fluid, bounded by the free surface $y=\eta(x, t)$ and the bottom $y=-h_0$, we have a velocity potential $\tilde{\Phi}(x, y, t)$ with $\boldsymbol{v}=\nabla\tilde{\Phi}$, satisfying the Laplace equation:

$$\tilde{\Phi}_{xx}+\tilde{\Phi}_{yy}=0\,. \tag{1.1}$$

On the free surface the kinematical condition should be satisfied:

$$\eta_t+\eta_x[\tilde{\Phi}_x]_{y=\eta}-[\tilde{\Phi}_y]_{y=\eta}=0\,, \tag{1.2}$$

and the condition of constant pressure:

$$g\eta+[\tilde{\Phi}_t+\tfrac{1}{2}(\tilde{\Phi}_x^2+\tilde{\Phi}_y^2)]_{y=\eta}=\text{constant }(=\gamma)\,. \tag{1.3}$$

At the bottom $y=-h_0$ holds the condition:

$$[\tilde{\Phi}_y]_{y=-h_0}=0\,. \tag{1.4}$$

We are interested in finding asymptotic solutions to this problem representing slowly varying wavetrains with wavelength of order unity. Taking K^{-1}, with $K\gg 1$, as a measure for the slow variations of amplitude, frequency, etc., it is useful to introduce stretched coordinates $x^*=x/K$, $t^*=t/K$. The vertical coordinate y is left undisturbed because the wave propagation is in the x- and t-direction only.

After omission of the asterisks the boundary value problem for $\tilde{\Phi}(x, y, t)$ is transformed into

$$\tilde{\Phi}_{xx}+K^2\tilde{\Phi}_{yy}=0\,,\ \text{for}\ \ -h_0<y<\eta(x, t) \tag{1.5}$$

$$[\tilde{\Phi}_y]_{y=-h_0}=0\,, \tag{1.6}$$

$$K[\tilde{\Phi}_t]_{y=\eta}+\tfrac{1}{2}[\tilde{\Phi}_x^2+K^2\tilde{\Phi}_y^2]_{y=\eta}+gK^2\eta=0\,, \tag{1.7}$$

$$K\eta_t+\eta_x[\tilde{\Phi}_x]_{y=\eta}-K^2[\tilde{\Phi}_y]_{y=\eta}=0\,, \tag{1.8}$$

where the Bernoulli constant γ has been absorbed into the potential $\tilde{\Phi}$.

2. Asymptotic Expansion with Respect to K

In terms of the stretched coordinates x and t each unit contains a large number (of order of magnitude K) wavelengths and periods respectively. Defining a phase function $S(x, t)$ by requiring that the derivative of $\tilde{\Phi}$ or η normal to a wavefront $S(x, t)=$constant is large of order K compared to the tangential derivative, we may introduce the following asymptotic expansion for a slowly varying wavetrain: (see also [3])

$$\tilde{\Phi}(x, y, t)=K\psi(x, t)+\Phi[KS(x, t), x, y, t]+\psi_1(x, t)+ + \frac{1}{K}\chi[KS(x, t), x, y, t]+\frac{1}{K}\psi_2(x, t)+O\left(\frac{1}{K^2}\right), \tag{2.1}$$

$$\eta(x, t)=U[KS(x, t), x, t]+\frac{1}{K}V[KS(x, t), x, t]+O\left(\frac{1}{K^2}\right), \tag{2.2}$$

with Φ, χ, U and V bounded functions for all values of x, y and t. The functions ψ, ψ_1 and ψ_2 account for the unbounded parts of $\tilde{\Phi}$ resulting from the presence of a mean velocity and from the absorption of the Bernoulli constant into $\tilde{\Phi}$. However, the derivatives of these unbounded terms should be bounded again for all values of x, y and t.

Defining the local wavenumber $\kappa=S_x$ and the local frequency $\omega=-S_t$ and introducing the notations:

$$p=KS(x, t)\,,\quad \beta=\psi_x\,,\quad \gamma=-\psi_t\,,\quad \beta_i=\psi_{ix}\,,\quad \gamma_i=-\psi_{it}\,,\quad (i=1, 2)$$

we insert asymptotic expansions (2.1) and (2.2) into boundary value problem (1.5)–(1.8) for $\tilde{\Phi}$ and η. Collecting powers of K and equating their coefficients to zero successively, the highest power of K yields a boundary value problem involving $\Phi(p, x, y, t)$, $U(p, x, t)$ and $\psi(x, t)$:

$$\kappa^2\Phi_{pp}+\Phi_{yy}=0\,, \qquad \text{for } -h_0<y<U(p, x, t)\,, \tag{2.3}$$

$$\Phi_y=0\,, \qquad \text{on } y=-h_0\,, \tag{2.4}$$

$$gU+(\beta\kappa-\omega)\Phi_p+\tfrac{1}{2}(\Phi_y^2+\kappa^2\Phi_p^2)=\gamma-\tfrac{1}{2}\beta^2\,, \qquad \text{on } y=U(p, x, t)\,, \tag{2.5}$$

$$(\beta\kappa-\omega)U_p+\kappa^2U_p\Phi_p-\Phi_y=0\,, \qquad \text{on } y=U(p, x, t)\,, \tag{2.6}$$

whereas the next power of K gives a boundary value problem containing the second order terms $\chi(p, x, y, t)$, $V(p, x, t)$ and $\psi_1(x, t)$:

$$\kappa^2\chi_{pp}+\chi_{yy}=-\kappa_x\Phi_p-2\kappa\Phi_{px}-\beta_x\,, \qquad \text{for } -h_0<y<U(p, x, t)\,, \tag{2.7}$$

$$\chi_y=0\,, \qquad \text{on } y=-h_0\,, \tag{2.8}$$

$$V[-\omega\Phi_{py}+(\beta+\kappa\Phi_p)\kappa\Phi_{py}+\Phi_y\Phi_{yy}+g]+\chi_p[-\omega+\kappa(\beta+\kappa\Phi_p)]+\Phi_y\,\chi_y+\beta\beta_1-\gamma_1= =-\Phi_t-(\beta+\kappa\Phi_p)\Phi_x\,, \qquad \text{on } y=U(p, x, t)\,, \tag{2.9}$$

$$V_p[-\omega+\kappa(\beta+\kappa\Phi_p)]+\kappa^2U_p\,\chi_p-\chi_y+V[\kappa^2U_p\Phi_{py}-\Phi_{yy}]+\beta_1\kappa U_p= =-U_t-\kappa U_p\Phi_x-(\beta+\kappa\Phi_p)U_x\,, \qquad \text{on } y=U(p, x, t)\,. \tag{2.10}$$

Note that in these problems the free surface condition has to be satisfied on the curve $y=U(p, x, t)$.

The boundary value problem for Φ, U and ψ is seen to be exactly identical with the problem for the uniform periodic wavetrain which is given in the Appendix by equation (A.2) and boundary conditions (A.3), (A.4) and (A.5) with $\kappa\theta$ replaced by p. The solution is given as two asymptotic power series in $\varepsilon=\kappa a$ for Φ and U, where a is a measure for the amplitude of the wave. Hence the leading terms Φ and U of the asymptotic expansion for the slowly varying wavetrain may be given up to terms of order ε^2:

$$\Phi(p, x, y, t) = A_1 \cosh[\kappa(h_0 + y)] \sin p + A_2 \cosh[2\kappa(h_0 + y)] \sin 2p + O(\varepsilon^3), \tag{2.11}$$

$$U(p, x, t) = b + a \cos p + a_2 \cos 2p + O(\varepsilon^3), \tag{2.12}$$

with $A_1 = O(\varepsilon)$, $A_2 = O(\varepsilon^2)$ and $a_2 = O(\varepsilon^2)$ defined by equations (A.14), (A.21) and (A.22) from the Appendix.

The leading terms of the asymptotic representation for slowly varying wavetrains represent locally a uniform wavetrain and contain six slowly varying parameters, viz. the wavenumber $\kappa(x, t)$ and frequency $\omega(x, t)$, which are of order unity, the amplitude $a(x, t)$, which is of order ε, and the mean waveheight $b(x, t)$, the mean velocity $\beta(x, t)$ and the pseudo-frequency $\gamma(x, t)$, which are of order ε^2. In the Appendix two relations between these quantities have been obtained, viz. dispersion relation (A.23):

$$\omega = \omega_0(\kappa) + \beta\kappa + \frac{gb\kappa^2}{2\omega_0 \cosh^2 \kappa h_0} + \tfrac{1}{2} g\kappa^3 a^2 \frac{D_0}{\omega_0}, \tag{2.13}$$

where

$$\omega_0^2(\kappa) = g\kappa \operatorname{tgh} \kappa h_0, \quad D_0 = \frac{9 \operatorname{tgh}^4 \kappa h_0 - 10 \operatorname{tgh}^2 \kappa h_0 + 9}{8 \operatorname{tgh}^3 \kappa h_0},$$

and pseudo-dispersion relation (A.24):

$$gb = \gamma - \frac{g\kappa a^2}{2 \sinh 2\kappa h_0}. \tag{2.14}$$

Furthermore two relations follow from the definition of κ, ω, β and γ:

$$\kappa_t + \omega_x = 0, \quad \beta_t + \gamma_x = 0. \tag{2.15a, b}$$

The two remaining equations will be obtained in the next section by imposing conditions of boundedness on the second terms $\chi(p, x, y, t)$ and $V(p, x, t)$ of the asymptotic representations (2.1) and (2.2) for slowly varying wavetrains.

3. The Boundary Value Problem Involving $\chi(p, x, y, t)$, $V(p, x, t)$ and $\psi_1(x, t)$

We will proceed by expanding χ and V in perturbation series with respect to ε and formulating the boundary value problems for the first and second order terms. In doing so we will meet secular terms that have to vanish in order to guarantee boundedness. This will provide the two remaining relations between the six slowly varying parameters of the problem.

We introduce the following expansions:

$$\chi(p, x, y, t) = \chi_1(p, x, y, t) + \chi_2(p, x, y, t) + O(\varepsilon^3),$$
$$V(p, x, t) = V_1(p, x, t) + V_2(p, x, t) + O(\varepsilon^3),$$

where the subscript 1 denotes terms of order ε and the subscript 2 terms of order ε^2. Furthermore we recall that Φ and U were expanded as follows:

$$\Phi(p, x, y, t) = \Phi_1(p, x, y, t) + \Phi_2(p, x, y, t) + O(\varepsilon^3),$$
$$U(p, x, t) = U_1(p, x, t) + U_2(p, x, t) + O(\varepsilon^3),$$

where the subscripts 1 and 2 again denote terms of order ε and ε^2 respectively. Noting that β, γ, β_1 and γ_1 are quantities of order ε^2, the above expansions are inserted into the boundary value problem for χ and V. For the terms of order ε the following problem is obtained:

$$\kappa^2\chi_{1pp}+\chi_{1yy}=-\kappa_x\Phi_{1p}-2\kappa\Phi_{1px}\,,\qquad \text{for } -h_0<y<0\,, \tag{3.1}$$

$$\chi_{1y}=0\,,\qquad \text{on } y=-h_0\,, \tag{3.2}$$

$$gV_1-\omega_0\chi_{1p}=-\Phi_{1t}\,,\qquad \text{on } y=0\,, \tag{3.3}$$

$$\omega_0V_{1p}+\chi_{1y}=U_{1t}\,,\qquad \text{on } y=0\,. \tag{3.4}$$

Notice that the expansion with respect to ε implies that the free surface conditions are to be satisfied on the line $y=0$, and that χ_1 and V_1 satisfy a linear, inhomogeneous boundary value problem.

For the second order terms χ_2 and V_2 we obtain the problem:

$$\kappa^2\chi_{2pp}+\chi_{2yy}=-\kappa_x\Phi_{2p}-2\kappa\Phi_{2px}-\beta_x\,,\qquad \text{for } -h_0<y<0\,, \tag{3.5}$$

$$\chi_{2y}=0\,,\qquad \text{on } y=-h_0\,, \tag{3.6}$$

$$gV_2-\omega_0\chi_{2p}=\omega_0V_1\Phi_{1py}+\omega_0U_1\chi_{1py}-\kappa^2\Phi_{1p}\chi_{1p}+\gamma_1+\\ -\Phi_{1y}\chi_{1y}-\Phi_{2t}-\kappa\Phi_{1p}\Phi_{1x}\,,\qquad \text{on } y=0\,, \tag{3.7}$$

$$\omega_0V_{2p}+\chi_{2y}=\kappa^2\Phi_{1p}V_{1p}+\kappa^2U_{1p}\chi_{1p}-U_1\chi_{1yy}-V_1\Phi_{1yy}+\\ +U_{2t}+\kappa U_{1p}\Phi_{1x}+\kappa\Phi_{1p}U_{1x}\,,\qquad \text{on } y=0\,. \tag{3.8}$$

At first let us consider the boundary value problem for χ_1 and V_1. Elimination of V_1 from free surface conditions (3.3) and (3.4) and using the explicit expressions for Φ_1 and U_1 yields the boundary condition for χ_1:

$$\frac{\omega_0^2}{g}\chi_{1pp}+\chi_{1y}=U_{1t}+\frac{\omega_0}{g}\Phi_{1pt}=\alpha_0\cos p\,,\quad \text{on } y=0\,, \tag{3.9}$$

with the abbreviation:

$$\alpha_0=a_t+\frac{\omega_0}{g}\frac{\partial}{\partial t}(A_1\cosh\kappa h_0)=\frac{\omega_0}{a}\frac{\partial}{\partial t}\left(\frac{a^2}{\omega_0}\right).$$

The differential equation for χ_1 becomes:

$$\kappa^2\chi_{1pp}+\chi_{1yy}=\alpha_1\cosh[\kappa(h_0+y)]\cos p+\alpha_2(h_0+y)\sinh[\kappa(h_0+y)]\cos p\,, \tag{3.10}$$

where we have put:

$$\alpha_1=-\kappa_xA_1-2\kappa A_{1x}\,,\quad \alpha_2=-2\kappa\kappa_xA_1\,,$$

and where A_1 is defined by equation (A.14).

A particular solution of equation (3.10) satisfying the bottom condition and which is bounded in the strip $-h_0\leqq y\leqq 0$, is given by the periodic function:

$$\chi_1^{(P)}=\frac{\alpha_2}{4\kappa}(y+h_0)^2\cosh[\kappa(h_0+y)]\cos p+\left(\frac{\alpha_1}{2\kappa}-\frac{\alpha_2}{4\kappa^2}\right)(y+h_0)\sinh[\kappa(h_0+y)]\cos p\,.$$

Putting $\chi_1=\chi_1^{(P)}+\tilde{\chi}_1$, then $\tilde{\chi}_1$ satisfies the equation

$$\kappa^2\tilde{\chi}_{1pp}+\tilde{\chi}_{1yy}=0\,,$$

with boundary conditions

$$[\tilde{\chi}_{1y}]_{y=-h_0}=0\quad\text{and}\quad\left[\frac{\omega_0^2}{g}\tilde{\chi}_{1pp}+\tilde{\chi}_{1y}\right]_{y=0}=\alpha\cos p\,,$$

where we have put

$$\alpha=\alpha_0+\frac{\omega_0^2}{g}\left(\frac{\alpha_1}{2\kappa}-\frac{\alpha_2}{4\kappa^2}\right)h_0\sinh\kappa h_0-\frac{\alpha_2}{2\kappa}h_0\cosh\kappa h_0+\\ -\left(\frac{\alpha_1}{2\kappa}-\frac{\alpha_2}{4\kappa^2}\right)(\sinh\kappa h_0+\kappa h_0\cosh\kappa h_0)\,.$$

It may be verified that this boundary value problem for $\tilde{\chi}_1$ has a solution of the form:

$$C\alpha\{p \sin p \cosh[\kappa(y+h_0)] - \kappa(y+h_0) \cos p \sinh[\kappa(h_0+y)]\},$$

with C a constant. This solution is not bounded for large p and hence the term $\alpha \cos p$ in the free surface condition for $\tilde{\chi}_1$ must vanish. This leads to the equation:

$$\alpha = \frac{\omega_0}{a}\frac{\partial}{\partial t}\left(\frac{a^2}{\omega_0}\right) - \kappa h_0 A_{1x}\frac{\sinh^2 \kappa h_0}{\cosh \kappa h_0} + A_1 h_0 \kappa_x \cosh \kappa h_0 + \\ + A_{1x}(\sinh \kappa h_0 + \kappa h_0 \cosh \kappa h_0) = 0 .$$

After considerable manipulation this equation reduces to the equation of conservation of "energy" $E = a^2/\omega_0$:

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}[\omega_0'(\kappa)E] = 0 . \tag{3.11}$$

For χ_2 we have the equation:

$$\kappa^2 \chi_{2pp} + \chi_{2yy} = -\beta_x - 2\kappa_x A_2 \cosh[2\kappa(h_0+y)] \cos 2p + \\ -4\kappa \frac{\partial}{\partial x}\{A_2 \cosh[2\kappa(h_0+y)]\} \cos 2p . \tag{3.12}$$

The inhomogeneous term of equation (3.12) contains terms with cos $2p$ and a term $-\beta_x$ which is constant with respect to p. The terms with cos $2p$ cause no difficulties but the constant term may induce unbounded terms in χ_2 as will appear below.

Furthermore we observe that the inhomogeneous terms in the free surface conditions (3.7) and (3.8) are made up by terms containing the factor cos $2p$ and terms that are constant with respect to p. Eliminating V_2 from conditions (3.7) and (3.8), it is clear that only the constant terms in the righthand side of condition (3.8) will remain in the resulting free surface condition for χ_2, because the constant terms of condition (3.7) vanish by differentiation with respect to p. Writing condition (3.8) as:

$$\omega_0 V_{2p} + \chi_{2y} = \kappa^2 \frac{\partial}{\partial p}(V_1 \Phi_{1p} + U_1 \chi_{1p}) + U_{2t} + \kappa U_{1p} \Phi_{1x} + \\ + \kappa \Phi_{1p} U_{1x} + U_1(\kappa_x \Phi_{1p} + 2\kappa \Phi_{1px}), \quad \text{on } y = 0 ,$$

it is seen that the constant term in the righthand side is independent of χ_1 and V_1. Schematically we have the following boundary value problem for χ_2:

$$\kappa^2 \chi_{2pp} + \chi_{2yy} = -\beta_x + [\ldots] \cos 2p , \qquad -h_0 < y < 0 ,$$

$$\chi_{2y} = 0 , \qquad \text{on } y = -h_0 ,$$

$$\frac{\omega_0^2}{g} \chi_{2pp} + \chi_{2y} = b_t + \tfrac{1}{2}\kappa a(A_{1x} \cosh \kappa h_0 + A_1 \kappa_x h_0 \sinh \kappa h_0) + \\ + \tfrac{1}{2}(\alpha\kappa)_x A_1 \cosh \kappa h_0 + [\ldots] \cos 2p , \qquad \text{on } y = 0 .$$

The terms with cos $2p$ do not give rise to unbounded terms in χ_2 and need not be considered further. If only the constant parts of the inhomogeneous terms are considered and a function $\tilde{\chi}_2$ is introduced by:

$$\chi_2 = -\tfrac{1}{2}\beta_x (y+h_0)^2 + \tilde{\chi}_2 ,$$

the resulting boundary value problem for $\tilde{\chi}_2$ is similar to that for $\tilde{\chi}_1$, only with the term α cos p in the free surface condition replaced by the constant term

$$\sigma = \beta_x h_0 + b_t + \tfrac{1}{2}\kappa a(A_{1x} \cosh \kappa h_0 + A_1 \kappa_x h_0 \sinh \kappa h_0) + \tfrac{1}{2}(a\kappa)_x A_1 \cosh \kappa h_0 .$$

Introducing the complex conjugate $\zeta(p, y)$ of $\tilde{\chi}_2(p, y)$ and putting

$$f(z)=\tilde{\chi}_2(p, y)+i\zeta(p, y)\,, \qquad z=\frac{p}{\kappa}+iy\,,$$

then $f(z)$ is an analytic function of z which has to be bounded in the strip $-h_0 \leqq \text{Im}\, z \leqq 0$. The free surface condition for $\tilde{\chi}_2$ may be written as:

$$\frac{\omega_0^2}{g}\tilde{\chi}_{2pp}+\tilde{\chi}_{2y}=\frac{\omega_0^2}{g}\tilde{\chi}_{2pp}-\kappa\zeta_p=\sigma\,, \quad \text{on } y=0\,.$$

Integration with respect to p then gives the condition:

$$\frac{\omega_0^2}{g}\tilde{\chi}_{2p}-\kappa\zeta=\sigma p+\text{constant}\,, \quad \text{on } y=0\,.$$

Now both χ_{2p} and ζ should be bounded for all p on $y=0$, so σp is a secular term. Hence $\sigma=0$, which leads to the equation:

$$\frac{\partial b}{\partial t}+h_0\frac{\partial \beta}{\partial x}+\frac{\partial}{\partial x}\left(\frac{ga^2\kappa}{2\omega_0}\right)=0\,. \tag{3.13}$$

This equation completes the set of six equations (2.13), (2.14), (2.15a, b), (3.11), (3.13) for the slowly varying functions $\kappa(x, t)$, $\omega(x, t)$, $a(x, t)$, $b(x, t)$, $\beta(x, t)$ and $\gamma(x, t)$. This set of equations may be reduced to a set of four equations by elimination of $\omega(x, t)$ and $\gamma(x, t)$:

$$\frac{\partial \kappa}{\partial t}+\frac{\partial}{\partial x}\left[\omega_0+\beta\kappa+\frac{gb\kappa^2}{2\omega_0\cosh^2\kappa h_0}+\tfrac{1}{2}g\kappa^3 D_0 E\right]=0\,,$$

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x}\left[\omega_0'(\kappa)E\right]=0\,,$$

$$\frac{\partial \beta}{\partial t}+\frac{\partial}{\partial x}\left[gb+\frac{g\kappa\omega_0 E}{2\sinh 2\kappa h_0}\right]=0\,,$$

$$\frac{\partial b}{\partial t}+\frac{\partial}{\partial x}\left[\beta h_0+\tfrac{1}{2}g\kappa E\right]=0\,,$$

which is in exact agreement with the results of Whitham [1], [2], and for a detailed investigation of these equations we refer to these two papers of Whitham.

Appendix: Periodic Progressive Stokes Waves

In this appendix we will derive the periodic progressive wavetrain solution of the original boundary value problem given by equation (1.1) together with boundary conditions (1.2), (1.3) and (1.4), which is expressed in the non-stretched coordinates x, y, t. Many authors ,starting with Stokes in 1847 have given solutions of this problem which are of the form:

$$\tilde{\Phi}(x, y, t)=\tilde{\Phi}(x-ct, y)\,, \quad \eta(x, t)=\eta(x-ct)\,,$$

where $\tilde{\Phi}$ and η are periodic functions of the phase $\theta=x-ct$. See for instance Lamb [4], Bowden [5] and Wehausen and Laitone [6]. In fact the assumption of periodicity of $\tilde{\Phi}$ in θ does not allow a mean velocity β in the x-direction. Such a mean velocity would give rise to a term βx in $\tilde{\Phi}$ which is not periodic in θ. In this Appendix a term βx is added to $\tilde{\Phi}$ and also the constant γ occurring in the condition of constant pressure on the free surface (equation 1.3) is absorbed into $\tilde{\Phi}$ by adding a term $-\gamma t$. Hence we will construct a solution of the problem of the following form:

$$\left.\begin{aligned}\tilde{\Phi}(x, y, t)&=\beta x-\gamma t+\varphi(x-ct, y)\\ \eta(x, t)&=\eta(x-ct)\end{aligned}\right\} \tag{A.1}$$

with φ and η periodic functions of the phase $\theta = x - ct$. This more general representation is also used by Whitham [1], [2], who proceeds by applying an averaged Lagrangian principle. In the present case we will use a perturbation series approach.

Insertion of equations (A.1) into the boundary value problem for $\tilde{\Phi}$ gives a boundary value problem involving $\varphi(\theta, y)$ and $\eta(\theta)$:

$$\varphi_{\theta\theta} + \varphi_{yy} = 0 , \qquad \text{for } -h_0 < y < \eta(\theta) , \tag{A.2}$$

$$\varphi_y = 0 , \qquad \text{on } y = -h_0 , \tag{A.3}$$

$$(\beta - c)\eta_\theta + \eta_\theta \varphi_\theta - \varphi_y = 0 , \qquad \text{on } y = \eta(\theta) , \tag{A.4}$$

$$g\eta + (\beta - c)\varphi_\theta + \tfrac{1}{2}(\varphi_\theta^2 + \varphi_y^2) = \gamma - \tfrac{1}{2}\beta^2 , \quad \text{on } y = \eta(\theta) . \tag{A.5}$$

An exact explicit solution of this problem is not available and all investigations of Stokes waves are concerned with finding series expansions of $\varphi(\theta, y)$ and $\eta(\theta)$ in powers of a small parameters $\varepsilon = \kappa a$, where κ is the wavenumber per 2π and a is a measure for the amplitude of the wave. This means that we are considering waves of moderately small amplitude and given wavelength $2\pi/\kappa$, and also we suppose that the various functions and constants occurring in the boundary value problem for $\varphi(\theta, y)$ and $\eta(\theta)$ can be expanded as follows:

$$\varphi(\theta, y) = \varepsilon\Phi_1(\theta, y) + \varepsilon^2\Phi_2(\theta, y) + \varepsilon^3\Phi_3(\theta, y) + \ldots,$$
$$\eta(\theta) = \varepsilon U_1(\theta) + \varepsilon^2 U_2(\theta) + \varepsilon^3 U_3(\theta) + \ldots,$$
$$c = \omega/\kappa = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \varepsilon^3 c_3 + \ldots,$$
$$\gamma = \varepsilon^2\gamma_0 + \varepsilon^3\gamma_1 + \ldots, \quad \beta = \varepsilon^2\beta_0 + \varepsilon^3\beta_1 + \ldots .$$

The first terms in the series for $\varphi(\theta, y)$ and $\eta(\theta)$ are assumed to be of order ε, corresponding to linear waves of infinitesimal amplitude. Furthermore, the constants β and γ, which are quantities that are absent in the linear theory, are assumed to be of order ε^2 and hence only play a role in the higher order perturbations accounting for the non-linear effects.

It is seen from equation (A.2) that all functions $\Phi_i(\theta, y)$ satisfy the equation:

$$\Phi_{i\theta\theta} + \Phi_{iyy} = 0 , \qquad i = 1, 2, 3, \ldots , \tag{A.6}$$

with boundary condition

$$\Phi_{iy} = 0 \quad \text{on } y = -h_0 . \tag{A.7}$$

The free surface conditions (A.4) and (A.5) are now expanded straightforwardly with respect to ε and yield the following conditions for the first order terms Φ_1 and U_1 to be satisfied on $y = 0$:

$$c_0 U_{1\theta} + \Phi_{1y} = 0 , \quad gU_1 - c_0\Phi_{1\theta} = 0 .$$

The second order terms Φ_2 and U_2 satisfy on $y = 0$:

$$c_0 U_{2\theta} + \Phi_{2y} = -c_1 U_{1\theta} + U_{1\theta}\Phi_{1\theta} - U_1\Phi_{1yy} , \tag{A.8}$$

$$gU_2 - c_0\Phi_{2\theta} = c_1\Phi_{1\theta} + c_0 U_1\Phi_{1\theta y} - \tfrac{1}{2}(\Phi_{1\theta}^2 + \Phi_{1y}^2) + \gamma_0 . \tag{A.9}$$

The third order terms Φ_3 and U_3 satisfy on $y = 0$:

$$c_0 U_{3\theta} + \Phi_{3y} = -c_1 U_{2\theta} + (\beta_0 - c_2)U_{1\theta} - U_1\Phi_{2yy} - U_2\Phi_{1yy} - \tfrac{1}{2}U_1^2\Phi_{1yyy} + + U_{2\theta}\Phi_{1\theta} + U_1 U_{1\theta}\Phi_{1\theta y} + U_{1\theta}\Phi_{2\theta} , \tag{A.10}$$

$$gU_3 - c_0\Phi_{3\theta} = c_0 U_1\Phi_{2\theta y} + \tfrac{1}{2}c_0 U_1^2\Phi_{1\theta yy} + c_0 U_2\Phi_{1\theta y} - \Phi_{1\theta}(U_1\Phi_{1\theta y} + \Phi_{2\theta}) + - \Phi_{1y}(U_1\Phi_{1yy} + \Phi_{2y}) + c_1(\Phi_{2\theta} + U_1\Phi_{1\theta y}) - (\beta_0 - c_2)\Phi_{1\theta} + \gamma_1 . \tag{A.11}$$

The first order problem for $\Phi_1(\theta, y)$ and $U_1(\theta)$ is solved by:

$$\varepsilon\Phi_1(\theta, y) = A_1 \cosh[\kappa(h_0 + y)] \sin \kappa\theta , \tag{A.12}$$

$$\varepsilon U_1(\theta) = a \cos \kappa\theta . \tag{A.13}$$

The unknown constants A_1 and c_0 are expressed in terms of a and κ by substitution of equations (A.12) and (A.13) into the free surface conditions for Φ_1 and U_1, yielding:

$$A_1 = \frac{ac_0}{\sinh \kappa h_0}, \tag{A.14}$$

and the dispersion relation for linear waves:

$$c_0^2 = \frac{g}{\kappa} \operatorname{tgh} \kappa h_0 . \tag{A.15}$$

The free surface conditions (A.8) and (A.9) for Φ_2 and U_2 now become respectively:

$$c_0 U_{2\theta} + \Phi_{2y} = c_1 \sin \kappa\theta - c_0 \operatorname{cotgh} \kappa h_0 \sin 2\kappa\theta , \tag{A.16}$$

$$gU_2 - c_0 \Phi_{2\theta} = c_0 c_1 \operatorname{cotgh} \kappa h_0 \cos \kappa\theta + \gamma_0 + \tfrac{1}{4} c_0^2 (1 - \operatorname{cotgh}^2 \kappa h_0) + \\ + \tfrac{1}{4} c_0^2 (3 - \operatorname{cotgh}^2 \kappa h_0) \cos 2\kappa\theta . \tag{A.17}$$

Elimination of $U_2(\theta)$ gives one condition for $\Phi_2(\theta, y)$ on $y=0$:

$$\frac{c_0^2}{g} \Phi_{2\theta\theta} + \Phi_{2y} = 2c_1 \sin \kappa\theta - c_0 \operatorname{cotgh} \kappa h_0 \sin 2\kappa\theta + \\ + \frac{c_0^3 \kappa}{g} \left(1 - \frac{1}{2 \sinh^2 \kappa h_0}\right) \sin 2\kappa\theta . \tag{A.18}$$

The general representation for $\Phi_2(\theta, y)$ having period $2\pi/\kappa$ and satisfying equation (A.6) with boundary condition (A.7) is:

$$\Phi_2(\theta, y) = \alpha_0 + \sum_{j=1}^{\infty} \alpha_j \cosh[j\kappa(y+h_0)] \sin j\kappa\theta . \tag{A.19}$$

Upon calculation of the unknown coefficients α_j by substitution of equation (A.19) into (A.18), it is seen that the term with $j=1$ automatically satisfies the homogeneous free surface condition

$$\frac{c_0^2}{g} \Phi_{2\theta\theta} + \Phi_{2y} = 0 \quad \text{on} \quad y = 0 ,$$

as a result of equation (A.15). Hence the term with $\sin \kappa\theta$ in the righthand side of (A.18) can never be produced by the periodic solution (A.19). So we put $c_1 = 0$. Homogeneous solutions of the form

$$\alpha_1 \cosh[\kappa(h_0 + y)] \sin \kappa\theta \tag{A.20}$$

can be added to $\Phi_2(\theta, y)$ of course, but defining the amplitude a by fixing $A_1 = ac_0/\sinh \kappa h_0$ as the coefficient of $\cosh[\kappa(h_0+y)] \sin \kappa\theta$ in the expansion of $\varphi(\theta, y)$, then terms such as (A.20) need not be taken into account in the higher order perturbations $\Phi_2(\theta, y)$, $\Phi_3(\theta, y)$, etc.

The solution of the second order problem is then:

$$\varepsilon^2 \Phi_2(\theta, y) = A_2 \cosh[2\kappa(h_0 + y)] \sin 2\kappa\theta ,$$

where A_2 is found by satisfying condition (A.18):

$$A_2 = \tfrac{3}{8} \frac{a^2 \omega_0}{\sinh^4 \kappa h_0} = O(\varepsilon^2) . \tag{A.21}$$

The corresponding second order approximation $\varepsilon^2 U_2(\theta)$ to the free surface elevation is calculated from equation (A.17):

$$\varepsilon^2 U_2(\theta) = b + a_2 \cos 2\kappa\theta ,$$

where the first approximation b to the mean height is given by

$$b = \frac{\varepsilon^2 \gamma_0}{g} - \frac{\kappa a^2}{2 \sinh 2\kappa h_0} = O(\varepsilon^2),$$

and a_2 is given by:

$$a_2 = \tfrac{1}{4}\kappa a^2 \frac{3 - \text{tgh}^2 \kappa h_0}{\text{tgh}^3 \kappa h_0} = O(\varepsilon^2). \tag{A.22}$$

In order to obtain the second order term $\varepsilon^2 c_2$ in the dispersion relation we have to investigate the secular terms, viz. terms containing $\sin \kappa\theta$, in the boundary condition for the third order problem. By substitution of the first and second order terms into conditions (A.10) and (A.11) the third order free surface conditions are found to be of the form:

$$c_0 U_{3\theta} + \Phi_{3y} = \mu_1 \sin \kappa\theta + \mu_3 \sin 3\kappa\theta,$$
$$g U_3 - c_0 \Phi_{3\theta} = \sigma_1 \cos \kappa\theta + \sigma_3 \cos 3\kappa\theta + \gamma_1,$$

with constants μ_1 and σ_1 given by:

$$\varepsilon^3 \mu_1 = -a^3\kappa^3(\beta_0 - c_2) - a\kappa^2 A_2 \cosh 2\kappa h_0 - \tfrac{3}{8}a^2\kappa^3 A_1 \sinh \kappa h_0 + \\ - \kappa^2 A_1(\tfrac{1}{2}a_2 + b)\cosh \kappa h_0,$$

$$\varepsilon^3 \sigma_1 = [2ac_0 A_2 \kappa^2 - \tfrac{1}{2}aA_1^2\kappa^3] \sinh 2\kappa h_0 + [\tfrac{3}{8}c_0 a^2 A_1 \kappa^3 - (\beta_0 - c_2)a^2\kappa^3 A_1] \cosh \kappa h_0 + \\ + [c_0 b A_1 \kappa^2 + \tfrac{1}{2}c_0 a_2 A_1 \kappa^2] \sinh \kappa h_0 - A_1 A_2 \kappa^2 \cosh 3\kappa h_0.$$

The constants μ_3 and σ_3 will not be needed because only terms with $\sin \kappa\theta$ and $\cos \kappa\theta$ will give rise to secular terms. Elimination of $U_3(\theta)$ gives a condition for $\Phi_3(\theta, y)$ only, which has to be satisfied on $y = 0$:

$$\frac{c_0^2}{g}\Phi_{3\theta\theta} + \Phi_{3y} = \left[\mu_1 + \frac{\sigma_1 \kappa c_0}{g}\right] \sin \kappa\theta + \left[\mu_3 + \frac{3\sigma_3 \kappa c_0}{g}\right] \sin 3\kappa\theta.$$

By a similar reasoning as before it follows that no periodic solution for $\Phi_3(\theta, y)$ can satisfy this condition with the term with $\sin \kappa\theta$ included in the righthand side and hence we have:

$$\mu_1 + \frac{\sigma_1 \kappa c_0}{g} = 0.$$

After considerable manipulation this relation reduces to:

$$\varepsilon^2(c_2 - \beta_0) = \frac{gb}{2c_0 \cosh^2 \kappa h_0} + \tfrac{1}{2}ga^2\kappa \frac{D_0}{c_0},$$

with the abbreviation:

$$D_0 = \frac{9\,\text{tgh}^4 \kappa h_0 - 10\,\text{tgh}^2 \kappa h_0 + 9}{8\,\text{tgh}^3 \kappa h_0}.$$

Summarizing, we have the following results up to the second order of approximation:

$$\tilde{\Phi}(x, y, t) = \beta x - \gamma t + A_1 \cosh[\kappa(h_0 + y)] \sin \kappa\theta + A_2 \cosh[2\kappa(h_0 + y)] \sin 2\kappa\theta + O(\varepsilon^3),$$
$$\eta(x, t) = b + a \cos \kappa\theta + a_2 \cos 2\kappa\theta + O(\varepsilon^3)$$

where the constants A_1, A_2 and a_2 are given by:

$$A_1 = \frac{ac_0}{\sinh \kappa h_0}, \quad A_2 = \tfrac{3}{8}\frac{a^2\omega_0}{\sinh^4 \kappa h_0},$$

$$a_2 = \tfrac{1}{4}\kappa a^2 \frac{3 - \text{tgh}^2 \kappa h_0}{\text{tgh}^3 \kappa h_0}.$$

Furthermore we have the dispersion relation

$$\omega(\kappa) = \kappa c = \omega_0(\kappa) + \beta\kappa + \frac{gb\kappa^2}{2\omega_0 \cosh^2 \kappa h_0} + \tfrac{1}{2}g\kappa^2 a^2 \frac{D_0}{c_0} + O(\varepsilon^3), \tag{A.23}$$

with $\omega_0^2(\kappa) = g\kappa \operatorname{tgh} \kappa h_0$, and also a "pseudo"-dispersion relation:

$$gb = \gamma - \frac{g\kappa a^2}{2 \sinh 2\kappa h_0} + O(\varepsilon^3)\,. \tag{A.24}$$

So it is seen that the uniform wavetrain solution depends on six parameters a, κ, ω, b, β and γ with two relations connecting them so that four of them may be chosen independently.

REFERENCES

[1] G. B. Whitham; Non-linear dispersion of water waves, *J. Fluid Mech.*, Vol. 27, pp. 399–412, 1967.

[2] G. B. Whitham; Variational methods and applications to water waves, *Proc. Roy. Soc. A*, 299, pp. 6–25, 1967.

[3] H. W. Hoogstraten; Dispersion of non-linear shallow water waves, *J. Engineering Math.*, Vol. 2, no. 3, pp. 249–273, 1968.

[4] H. Lamb; *Hydrodynamics*, Cambridge U.P. 1932/Dover Press, 1945.

[5] K. F. Bowden; Some observations of waves and other fluctuations in a tidal current; Appendix: Surface waves of finite amplitude in water of finite depth, *Proc. Roy. Soc., A*, 192, p. 403, 1948.

[6] J. V. Wehausen and E. Laitone; Surface Waves, in: *Handbuch der Physik, Strömungslehre*, Band IX, Springer Verlag, Heidelberg, 1960.

Journal of Engineering Math., Vol. 3 (1969) 45–54